Lecture-8

Uniqueness theorem

Uniqueness theorem

Since the Poisson's and Laplace' equation are second order PDEs, solution of each of them need two boundary conditions specifying the value of potentials at two different points on the boundary.

<u>Statement</u> : The solution of Laplace's (or Poisson's) equations is unique if it satisfies Laplace's equation and also the boundary conditions.

<u>Proof</u>:

Assume that the solution is not unique; i.e., let V_1 and V_2 be the different solutions of the Laplace' equation, V_1 and V_2 being general functions of the coordinates used.

This means $$\nabla^2 V_1 = 0 \quad and \quad \nabla^2 V_2 = 0$$

and therefore $$\nabla^2 (V_1 - V_2) = 0$$

Each of the solutions V_1 and V_2 must satisfy the boundary conditions. Let the given (known) values on the boundaries be V_b , then V_{1b} will be the vaule of V_1 at the boundary and V_{2b} be the value of V_2 at the boundary. Therefore V_1 and V_2 must be identical to V_b.

$$V_{1b} = V_{2b} = V_b \qquad or \qquad V_{1b} - V_{2b} = 0 \qquad \text{--- (9)}$$

We have the vector identity

$$\nabla \cdot (\varphi \vec{A}) = \varphi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \varphi)$$

where A is a vector and ϕ is a scalar.

Now letting the scalar $\phi = (V_1 - V_2)$ and the vector
A $= \nabla(V_1 - V_2)$, we get

$$\nabla \cdot [(V_2 - V_1)\nabla(V_2 - V_1)] = (V_2 - V_1)[(\nabla \cdot \nabla(V_2 - V_1)) + \nabla(V_2 - V_1) \cdot \nabla(V_2 - V_1)]$$

Now, Let us integrate both sides throughout the volume enclosed by the boundary surface chosen to get

$$\int_{vol} \nabla \cdot [(V_2 - V_1)\nabla(V_2 - V_1)]dv = \int_{vol} (V_2 - V_1)[(\nabla \cdot \nabla(V_2 - V_1)]\,dv$$

$$+ \int_{vol} [\nabla(V_2 - V_1)]^2\,dv \qquad \text{--- (10)}$$

Using divergence theorem the volume integral on the LHS of equation (10) can be converted into a surface integral over the surface surrounding the volume ; this surface consists of the boundaries already specified on which $V_{1b} - V_{2b} = 0$ to yield

$$\int_{vol} \nabla \cdot [(V_2 - V_1)\nabla(V_2 - V_1)]dv = \oint_S [(V_{2b} - V_{1b})\nabla(V_{2b} - V_{1b})] \cdot d\vec{S} = 0$$

One of the factors in the first integral on the RHS of equation (10) is zero by hypothesis. Therefore that integral is zero. Therefore equation (10) reduces to

$$\int_{vol} [\nabla (V_{`2} - V_1)]^2 \, dv = 0$$

In this equation, either $V_2 - V_1$ must be zero or dv the volume element must be zero. dv is arbitrary and hence cannot be zero. Therefore $V_2 - V_1$ must be zero. Or $V_1 = V_2$.

Also, if the gradient of $V_2 - V_1$ is every where is zero, then $V_2 - V_1$ cannot change with any coordinates. Therefore, $V_2 - V_1$ = constant. At the boundary , from equation (9), the Constant turns out to be zero. i.e., $V_{1b} - V_{2b} = 0 = V_2 - V_1$

Therefore we get $V_2 = V_1$ meaning the two solutions are Identical.